

Conjectures and questions in convex geometry

of interest for quantum theory
and other physical statistical theories

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Abstract: Some conjectures and open problems in convex geometry are presented, and their physical origin, meaning, and importance, for quantum theory and generic statistical theories, are briefly discussed.

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1 Introduction

In this note I present a couple of conjectures and open problems in convex geometry, wishing that they will raise the interest of geometers and be solved soon.

What is the origin of these conjectures and open problems? There is a branch of Bayesian probability theory, called the theory of statistical models, that studies the probabilistic relations among particular sets of propositions or variables [1–4, and refs therein]. Its range of applications is therefore as vast and diverse as that of probability theory. One of these applications, which is gaining the interest of more and more researchers in physics, mathematics, and statistics, is the study of the probabilistic and communication-theoretic features of quantum theory and other physical statistical theories, and of how these features can be mimicked by or emerge from a classical theory.

Convex geometry is one of the main mathematical structures at the core of the theory of statistical models. So with this theory we can translate some theorems and conjectures about quantum and

non-quantum theories and their relations with classical theories into theorems and conjectures about convex geometry, and vice versa. It is these physical conjectures that are here presented, translated in strictly convex-geometric terms.

The presentation follows standard notation and terminology [5–12]. Some definitions are presented in the next section, in particular the notion of a statistical model; then the notion of *refinement* of a statistical model is presented in § 3 with many illustrative examples: this is the notion to which are connected the open problems and conjectures presented in § 4. Physical motivation and meaning are discussed in an appendix.

2 Definitions

In this section we consider a compact convex space \mathcal{C} of dimension n .

Definition 1. The *convex-form space* $\mathcal{P}_{\mathcal{C}}$ of the convex space \mathcal{C} is the set of all affine forms on \mathcal{C} with range in $[0, 1]$, called *convex forms*:

$$\mathcal{P}_{\mathcal{C}} := \{v: \mathcal{C} \rightarrow [0, 1] \mid v \text{ is affine}\}. \quad (1)$$

Convex combination is easily defined on this set, which is thus a convex space itself. A vector sum and difference can also be naturally defined but the set is not closed under them. The forms $v_0: a \mapsto 0$ and $v_u: a \mapsto 1$ are called *null-form* and *unit-form*. The action of a form v on a point a is denoted by $v \cdot a$. We call v an *extreme form* if it is an extreme point of $\mathcal{P}_{\mathcal{C}}$.

Later on we shall on occasion write vector differences of convex forms when their result is still a convex form.

It is useful to recall that a non-constant convex form v on \mathcal{C} is determined by an ordered pair of $(n - 1)$ -dimensional, parallel hyperplanes non-intersecting the interior of \mathcal{C} , as explained in fig. 1.

Fact 1. If the convex space \mathcal{C} has dimension n then its convex-form space $\mathcal{P}_{\mathcal{C}}$ has dimension $n + 1$. If \mathcal{C} is a polytope, so is $\mathcal{P}_{\mathcal{C}}$. The convex structure of $\mathcal{P}_{\mathcal{C}}$ is determined by that of \mathcal{C} , and its affine span $\text{aff } \mathcal{P}_{\mathcal{C}}$ is the space of affine forms on $\text{aff } \mathcal{C}$. $\mathcal{P}_{\mathcal{C}}$ is a bi-cone whose vertices are the null-form v_0 and the unit-form v_u , and this bi-cone is centro-symmetric with centre of symmetry $(v_0 + v_u)/2$. The

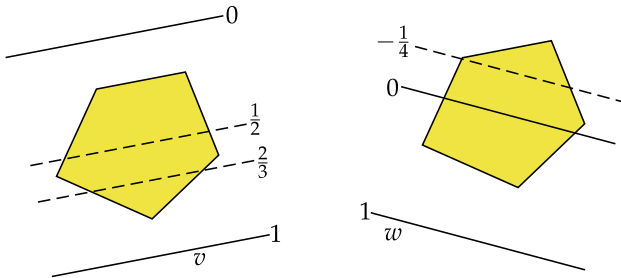


Figure 1: The contour hypersurfaces of an affine form are parallel hyperplanes; such a form is completely determined by assigning the two hyperplanes corresponding to the values 0 and 1. To be a convex form, these two hyperplanes must not intersect the interior of the convex space on which the form is defined. On the left, v is a convex form for the pentagonal convex space; two lines are indicated where v has values $1/2$ and $2/3$; no points of the space yield the values 0 or 1. On the right, w cannot be a convex form (although it is an affine form) because it assigns strictly negative values to some points of the convex space; this happens because the 0-value line cuts the convex space.

number of extreme points of $\mathcal{P}_{\mathcal{C}}$ besides v_0 and v_u is determined by the structure of the faces of \mathcal{C} ; e.g., if \mathcal{C} is a two-dimensional polytope, the number of extreme points of $\mathcal{P}_{\mathcal{C}}$ equals $2m + 2$, where m is the number of bounding directions of \mathcal{C} .

For example, the convex-form space of an n -dimensional simplex is an $(n + 1)$ -dimensional parallelotope (with 2^{n+1} extreme points), and that of a parallelogram is an octahedron, as shown in fig. 2; that of a pentagon is a pentagonal trapezohedron, shown in fig. 6.

The following definition introduces the most important mathematical objects of our study:

Definition 2. A statistical model is a pair $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ of a convex polytope and its convex-form space; its *dimension* is simply the dimension of \mathcal{C} . A *simplicial* statistical model is one in which \mathcal{C} is an n -dimensional simplex Δ , and therefore \mathcal{P}_{Δ} is an $(n + 1)$ -dimensional parallelotope.

The name, especially the adjective ‘statistical’, is admittedly sibylline, but it rightly prophesies a connexion with probability theory.

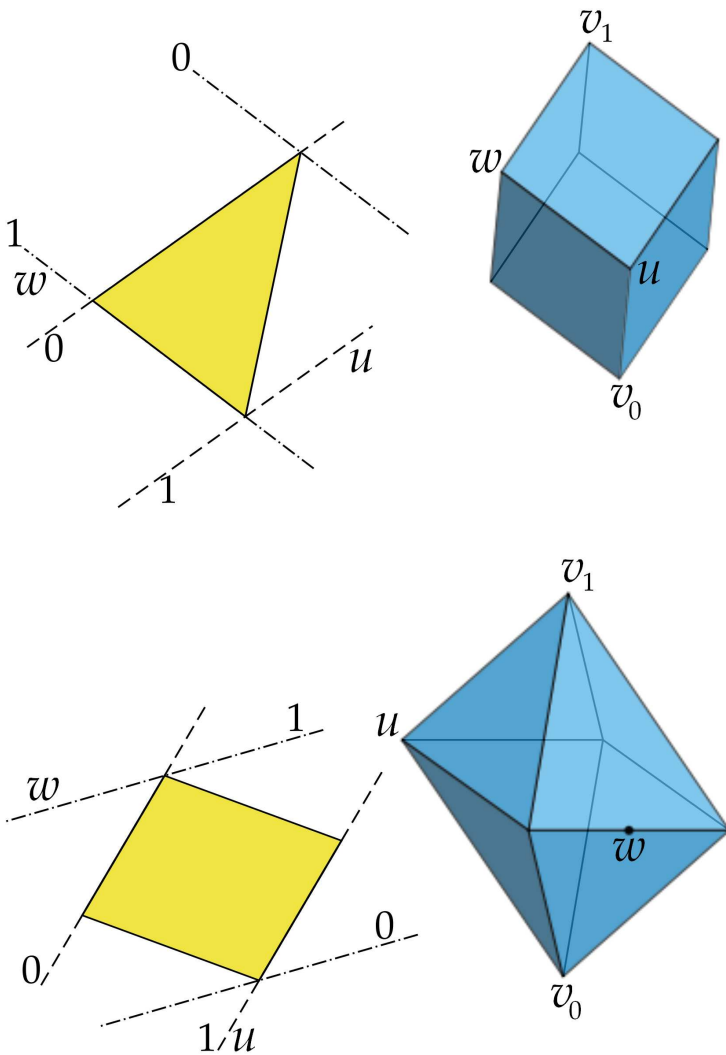


Figure 2: Two two-dimensional convex spaces, on the left, with their three-dimensional convex-form spaces, on the right. They constitute two statistical models. The convex forms w , u are represented as pairs of parallel lines on the convex spaces and as points on the convex-form spaces. v_0 and v_1 are the null- and unit-forms.

3 Refinement of a statistical model

In this section we consider compact convex *polytopes* of dimension n ; i.e., we are assuming that our convex spaces have a finite number of extreme points.

In convex geometry it is a well-known fact that any polytope can be obtained as a section or a projection of a usually higher-dimensional simplex (Grünbaum [8], § 5.1, theorems 1 and 2). Both these kinds of ‘correspondence’ $\mathcal{C} \longleftrightarrow \Delta$ between a polytope \mathcal{C} and the simplex Δ from which the polytope is obtained have the following characteristics:

- C1. each point of \mathcal{C} has at least one corresponding point in Δ , i.e., the correspondence $\mathcal{C} \longleftrightarrow \Delta$ is defined on all \mathcal{C} ;
- C2. there may be points of Δ with no corresponding points in \mathcal{C} , i.e., the correspondence $\mathcal{C} \longleftrightarrow \Delta$ needs not be defined on all Δ ;
- C3. several points in Δ can correspond to one and the same point in \mathcal{C} , i.e., $\mathcal{C} \longleftrightarrow \Delta$ can be one-to-many;
- C4. at most one point in \mathcal{C} can correspond to one in Δ , i.e., $\mathcal{C} \longleftrightarrow \Delta$ cannot be many-to-one;
- C5. a convex combination of points in Δ corresponds to the same convex combination of the corresponding points in \mathcal{C} , when the latter are defined, i.e., $\mathcal{C} \longleftrightarrow \Delta$ is affine.

These characteristics mathematically pin down the $\mathcal{C} \longleftrightarrow \Delta$ correspondence as a partial, onto, affine map from Δ to \mathcal{C} (we denote partial maps by hooked arrows):

$$F : \Delta \xrightarrow{\text{onto, affine}} \mathcal{C}, \quad (2)$$

- ‘onto’ = it covers \mathcal{C} , from C1;
- ‘partial’ = it needs not be defined on all Δ , from C2;
- ‘affine’ = if F is defined on a, b then $F[\lambda a + (1 - \lambda)b] = \lambda F(a) + (1 - \lambda)F(b)$, from C5.
- ‘map’ = it is many-to-one or one-to-one, but not one-to-many, from C3 and C4.

Intuitively, this map says that the simplex Δ has a ‘finer’ structure than the polytope \mathcal{C} , or that \mathcal{C} is a ‘coarser’ image of Δ because parts of the latter are either missing or not distinguishable in \mathcal{C} . We might

call Δ a ‘simplicial refinement’ of \mathcal{C} . The projection or section of a simplex are particular cases of this map.

It is natural to try to generalize this kind of construction and its associated theorems from polytopes to statistical models: given a statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$, one asks whether it can be obtained from a simplicial statistical model $(\Delta, \mathcal{P}_{\Delta})$, considered as a ‘refinement’.

More precisely, we want a correspondence $\mathcal{C} \rightsquigarrow \Delta$ between the convex spaces, one $\mathcal{P}_{\mathcal{C}} \rightsquigarrow \mathcal{P}_{\Delta}$ between their convex-form spaces, and we want both correspondences to satisfy requirements analogous to C1–C5. Moreover, we clearly want these correspondences to preserve the action of convex forms on the respective convex spaces in the two statistical models.

This means that the two correspondences have to be expressed by partial, surjective, affine maps from Δ to \mathcal{C} and from \mathcal{P}_{Δ} to $\mathcal{P}_{\mathcal{C}}$

$$F : \Delta \xrightarrow{\text{onto, affine}} \mathcal{C}, \quad (3)$$

$$G : \mathcal{P}_{\Delta} \xrightarrow{\text{onto, affine}} \mathcal{P}_{\mathcal{C}} \quad (4)$$

that satisfy

$$G(v) \cdot F(a) = v \cdot a \quad \text{for all } a \in \Delta, v \in \mathcal{P}_{\Delta} \text{ on which } F, G \text{ are defined.} \quad (5)$$

We are thus led to the following

Definition 3. A *simplicial refinement* of a statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ is a set $(\Delta, \mathcal{P}_{\Delta}, F, G)$ where:

- I. $(\Delta, \mathcal{P}_{\Delta})$ is a simplicial statistical model,
- II. $F : \Delta \hookrightarrow \mathcal{C}$ is partial, onto, and affine,
- III. $G : \mathcal{P}_{\Delta} \hookrightarrow \mathcal{P}_{\mathcal{C}}$ is partial, onto, and affine,
- IV. F and G are such that $G(v) \cdot F(a) = v \cdot a$ on their domains of definition.

We shall often omit the adjective ‘simplicial’ when speaking about a simplicial refinement.

Example 1. Consider the statistical model where \mathcal{C} is a parallelogram and $\mathcal{P}_{\mathcal{C}}$ an octahedron, as at the bottom of fig. 2 or 3. A simplicial

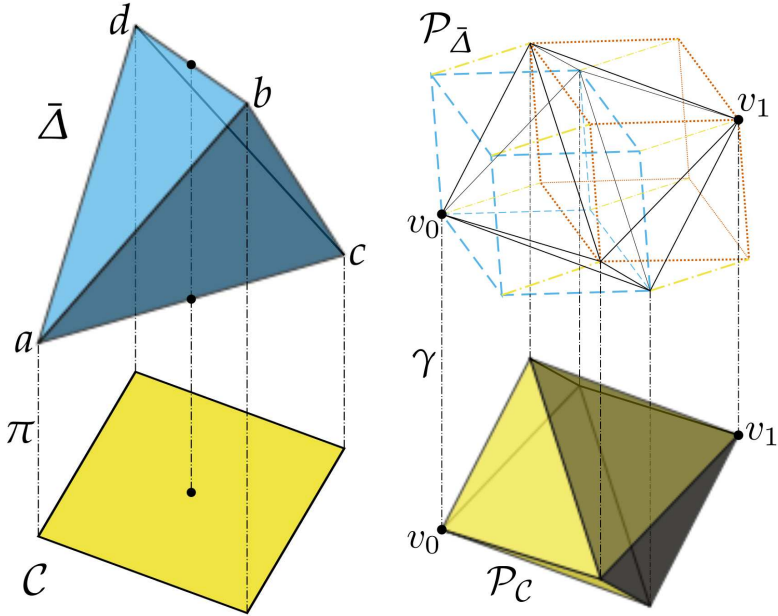


Figure 3: Illustration of the refinement of the statistical model of Example 1. Note how the mapping $\pi: \bar{\Delta} \rightarrow \mathcal{C}$ is total and non-injective, and $\gamma: \mathcal{P}_{\bar{\Delta}} \hookrightarrow \mathcal{P}_{\mathcal{C}}$ is partial; both are projections. The representation of the hypercube or four-dimensional parallelotope $\mathcal{P}_{\bar{\Delta}}$ is explained in fig. 4 on the next page.

refinement is given by $(\bar{\Delta}, \mathcal{P}_{\bar{\Delta}}, \pi, \gamma)$, where: $\bar{\Delta}$ is a tetrahedron; $\mathcal{P}_{\bar{\Delta}}$ a hypercube; the map π is the projection of the tetrahedron onto the parallelogram; the map γ maps the zero- and unit-forms of $\mathcal{P}_{\bar{\Delta}}$ onto those of $\mathcal{P}_{\mathcal{C}}$, while the other four extreme forms of $\mathcal{P}_{\mathcal{C}}$ are the images of the following forms on $\bar{\Delta}$ in the notation of fig. 3:

- that having zero value on the vertices a and b and unit value on c and d ,
- as the previous but with zero and unit values exchanged,
- that having zero value on the vertices a and d and unit value on b and c ,
- as the previous but with zero and unit values exchanged.

We see that π is a total map, defined on all $\bar{\Delta}$; whereas γ is partial: in particular, it is not defined on the non-constant extreme forms of $\mathcal{P}_{\bar{\Delta}}$. ♣

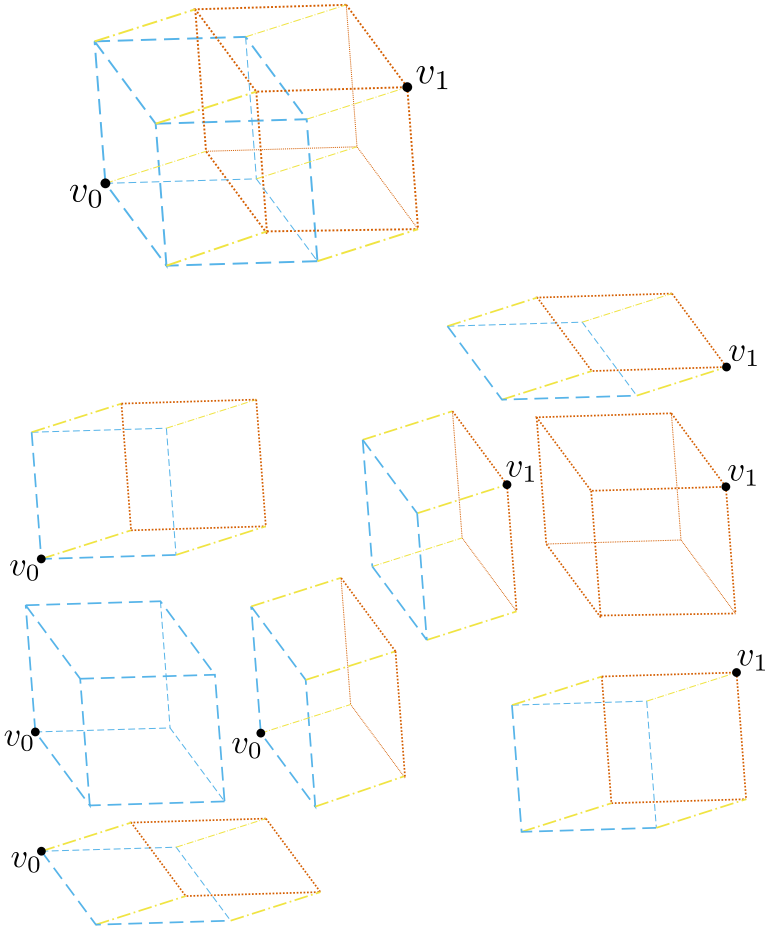


Figure 4: A hypercube, or four-dimensional parallelotope, is a four-dimensional polytope with eight, pairwise parallel, three-dimensional facets, all parallelepipeds; and 24 two-dimensional faces, all parallelograms. The figure on the top is the projection of a 4-parallelotope onto a three-dimensional space (further projected on paper); because of the dimensional reduction some of the projected facets intersect each other. To help you distinguish all eight of them in the top figure, they are separately represented underneath it. As the convex-form space of a three-dimensional simplex (tetrahedron), two vertices of the hypercube represent the nought- and unit-forms, also indicated in the figure.

The preceding example is based on the fact that any convex polytope with m extreme points can be obtained as the *projection* of an $(m - 1)$ -dimensional simplex: see again Grünbaum [8], § 5.1, theorem 2. We have the following

Fact 2. The theorem just mentioned can always be used to construct a simplicial refinement, in the guise of Example 1, of *any* statistical model; cf. Holevo [2, § I.5].

We already said that another theorem of convex geometry states that any convex polytope with m facets can be obtained as the *section* of an $(m - 1)$ -dimensional simplex [8, § 5.1, theorem 1]. This theorem *cannot* be used to construct a simplicial refinement, though, as shown in the following

Counter-example 2. The parallelogram \mathcal{C} of the preceding example can be obtained as the intersection of the tetrahedron $\bar{\Delta}$ and an intersecting plane parallel to the segments ac and bd of fig. 5. This defines a partial, surjective, affine map $F: \bar{\Delta} \hookrightarrow \mathcal{C}$, which is simply the identity, $F(s) = s$, in its domain of definition $\mathcal{C} \subset \bar{\Delta}$. However, it is impossible to find an affine map $G: \mathcal{P}_{\bar{\Delta}} \hookrightarrow \mathcal{P}_{\mathcal{C}}$ that be surjective and such that $v \cdot s = G(v) \cdot F(s) \equiv G(v) \cdot s$: the extreme forms of $\mathcal{P}_{\mathcal{C}}$, for example, cannot have any counter-image. The reason for this is geometrically explained in fig. 5. Thus we cannot construct a simplicial refinement of $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ where \mathcal{C} is obtained by sectioning $\bar{\Delta}$. ✎

Linusson [13] has shown that the previous counter-example is generally valid:

Theorem 1 (Linusson). *Given a non-simplicial statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ it is impossible to find a simplicial refinement $(\Delta, \mathcal{P}_{\Delta}, F, G)$ such that F is injective in its domain of definition (which means that F would represent the intersection of Δ with a hyperplane, whereby \mathcal{C} is obtained).*

Example 3. Let \mathcal{C} be a two-dimensional pentagonal convex space with extreme points $\{s_1, s_2, s_3, s_4, s_5\}$. Its convex-form space $\mathcal{P}_{\mathcal{C}}$ is a pentagonal trapezohedron that has, besides the null- and unit-forms v_0 and v_u , ten other extreme points given by the forms

$$\{v_1, \dots, v_5, v_u - v_1, \dots, v_u - v_5\}$$

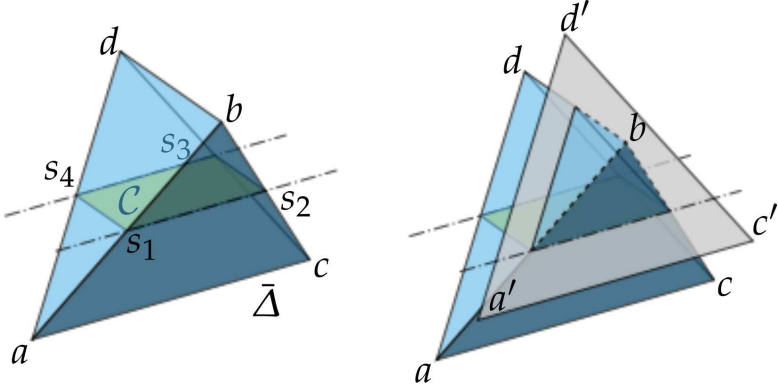


Figure 5: An extreme convex form v of the two-dimensional parallelogram \mathcal{C} is represented by the two parallel dot-dashed lines s_1s_2 and s_3s_4 lying in the same plane as \mathcal{C} (left figure). A convex form w of $\bar{\Delta}$ is represented by two parallel planes, and if w is to correspond to v , $G(w) = v$, in such a way that $G(w) \cdot s_i = v \cdot s_i$, these two planes must contain s_1s_2 and s_3s_4 ; they cannot intersect the interior of $\bar{\Delta}$, however, if w is to be a convex form. But it is impossible to satisfy both requirements for both planes: e.g., the only plane that contains the line s_3s_4 and does not cut $\bar{\Delta}$ is acd (right figure); then $a'c'd'$ is the parallel plane containing s_1s_2 , but this plane cuts $\bar{\Delta}$ (meaning, e.g., that $w \cdot b < 0$ or $w \cdot b > 1$). All other constructions one can think of have the same problem. Thus the map G cannot exist.

such that

$$(v_i \cdot s_j) = \begin{pmatrix} 1 & \alpha & 0 & 0 & \alpha \\ \alpha & 1 & \alpha & 0 & 0 \\ 0 & \alpha & 1 & \alpha & 0 \\ 0 & 0 & \alpha & 1 & \alpha \\ \alpha & 0 & 0 & \alpha & 1 \end{pmatrix} \quad \text{with } \alpha := \frac{\sqrt{5}-1}{2}, \quad (6)$$

$$(v_u - v_i) \cdot s_j = v_u \cdot s_j - v_i \cdot s_j = 1 - v_i \cdot s_j;$$

see fig. 6.

A refinement of this model is given by $(\Delta, \mathcal{P}_\Delta, F, G)$ where Δ is a nine-dimensional simplex, or decatope, with ten extreme points $\{e_1, \dots, e_{10}\}$, and \mathcal{P}_Δ is a ten-dimensional parallelotope with twelve extreme points given by the forms

$$\{d_0, d_u, d_1, \dots, d_{10}, d_u - d_1, \dots, d_u - d_{10}\}$$

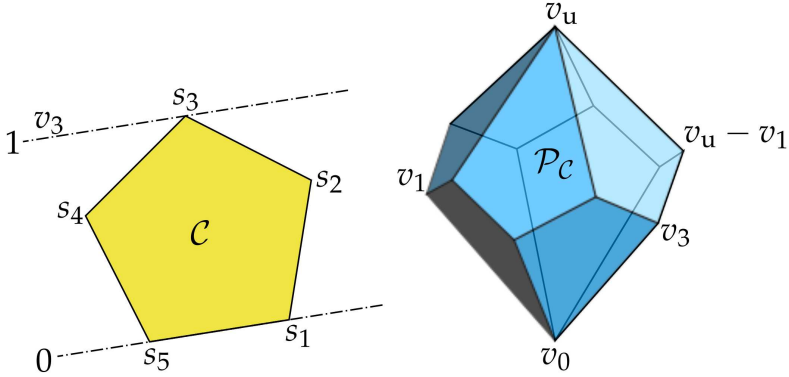


Figure 6: The pentagonal convex set \mathcal{C} and its convex-form space $\mathcal{P}_{\mathcal{C}}$ from Example 3. The convex form v_3 is shown on \mathcal{C} (as a pair of parallel lines) and on $\mathcal{P}_{\mathcal{C}}$ (as a point).

such that

$$d_0 \cdot e_j = 0, \quad d_u \cdot e_j = 1, \quad d_i \cdot e_j = \delta_{ij}, \quad (d_u - d_i) \cdot e_j = 1 - \delta_{ij}. \quad (7)$$

The map F is defined on the points

$$\begin{aligned} F[(e_1 + e_2)/2] &= s_1, & F[(e_3 + e_4)/2] &= s_2, & F[(e_5 + e_6)/2] &= s_3, \\ F[(e_7 + e_8)/2] &= s_4, & F[(e_9 + e_{10})/2] &= s_5, \end{aligned} \quad (8)$$

and their convex combinations; i.e., it is partial and defined on a four-dimensional simplex given by the convex span of $\{(e_1 + e_2)/2, (e_3 + e_4)/2, \dots, (e_9 + e_{10})/2\}$; see fig. 7.

The map G is also partial: define the following one-dimensional convex subsets

$$\begin{aligned} \Theta_1 &:= \{d_1 + d_2 + x(d_9 + d_3) + y_x(d_{10} + d_4) \mid x \in [2\alpha - 1, 1]\}, \\ \Theta_2 &:= \{d_3 + d_4 + x(d_1 + d_5) + y_x(d_2 + d_6) \mid x \in [2\alpha - 1, 1]\}, \\ \Theta_3 &:= \{d_5 + d_6 + x(d_3 + d_7) + y_x(d_4 + d_8) \mid x \in [2\alpha - 1, 1]\}, \\ \Theta_4 &:= \{d_7 + d_8 + x(d_5 + d_9) + y_x(d_6 + d_{10}) \mid x \in [2\alpha - 1, 1]\}, \\ \Theta_5 &:= \{d_9 + d_{10} + x(d_7 + d_1) + y_x(d_8 + d_2) \mid x \in [2\alpha - 1, 1]\}, \end{aligned} \quad (9)$$

with $y_x := 2\alpha - x$;

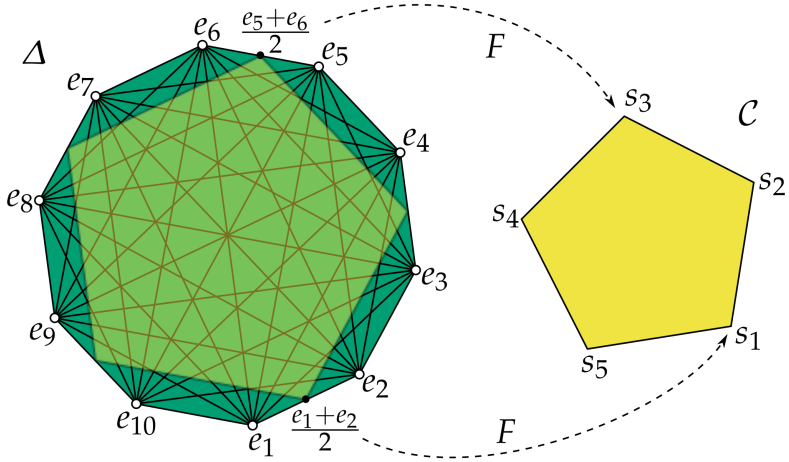


Figure 7: Representation of the decatope Δ and the map F of Example 3. Δ is represented by its graph [8, §§ 11.3, 8.4], which can be understood as a parallel projection of Δ onto a two-dimensional plane: the vertices represent its extreme points $\{e_i\}$, the $\binom{10}{2} = 45$ lines connecting any two vertices represent its edges, and all $\binom{10}{3} = 120$ triangles connecting any three vertices represent its two-dimensional faces. The map F is only defined on a four-dimensional, simplicial convex subset (in lighter green) of Δ .

then G is defined by

$$\begin{aligned} G(d_0) &= v_0, & G(d_u) &= v_u, & G(\Theta_i) &= \{v_i\}, \quad i = 1, \dots, 5, \\ G(d_u - \Theta_i) &= \{v_u - v_i\}, \quad i = 1, \dots, 5, \end{aligned} \quad (10)$$

where $d_u - \Theta_i := \{d_u - a \mid a \in \Theta_i\}$.

The important features of this refinement are these:

- each extreme point of \mathcal{C} corresponds to a *non-extreme* point of Δ , and to that alone;
- each of the ten non-trivial extreme points of $\mathcal{P}_{\mathcal{C}}$ corresponds to a non-zero-dimensional convex set of non-extreme points of \mathcal{P}_{Δ} ;
- the map F is partial, i.e. it is not defined on some points of Δ , not even its ten pure ones $\{e_i\}$; contrast this with Example 1;

- d. on the four-dimensional simplex on which it is defined, the map F acts as a projection onto \mathcal{C} , analogously to the map π of Example 1. ✎

In the example just discussed, the fact that the extreme points of \mathcal{C} correspond to single points of Δ makes it possible for the extreme points of $\mathcal{P}_{\mathcal{C}}$ to correspond to one-dimensional sets in \mathcal{P}_{Δ} . But the opposite situation is also possible:

Example 4. The sets \mathcal{C} , $\mathcal{P}_{\mathcal{C}}$, Δ , \mathcal{P}_{Δ} are defined as in the preceding example, but the maps F and G are defined differently. Consider these five one-dimensional faces of Δ :

$$\begin{aligned} E_1 &:= \text{conv}\{e_1, e_2\} = \{xe_1 + (1-x)e_2 \mid x \in [0, 1]\} \\ E_2 &:= \text{conv}\{e_3, e_4\}, & E_3 &:= \text{conv}\{e_5, e_6\}, \\ E_4 &:= \text{conv}\{e_7, e_8\}, & E_5 &:= \text{conv}\{e_9, e_{10}\}; \end{aligned} \quad (11)$$

then define F by

$$F(E_i) = \{s_i\}, \quad i = 1, \dots, 5, \quad (12)$$

see fig. 8, and G by

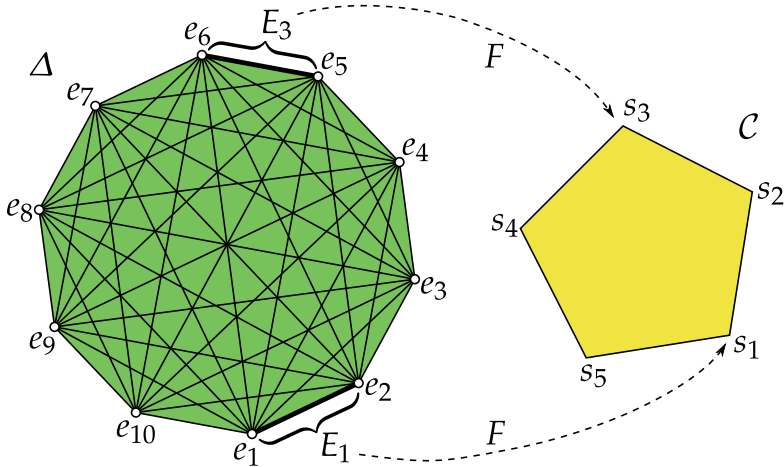


Figure 8: Representation of the decatope Δ and the map F of Example 4; cf. fig. 7. The map F now maps five edges of Δ to the $\{s_i\}$ and by convex combination is defined on all of Δ .

$$\begin{aligned}
G(d_0) &= v_0, & G(d_u) &= v_u, \\
G[d_1 + d_2 + \alpha(d_3 + d_9) + \alpha(d_4 + d_{10})] &= r_1, \\
G[d_3 + d_4 + \alpha(d_1 + d_5) + \alpha(d_2 + d_6)] &= r_2, \\
G[d_5 + d_6 + \alpha(d_3 + d_7) + \alpha(d_4 + d_8)] &= r_3, \\
G[d_7 + d_8 + \alpha(d_5 + d_9) + \alpha(d_6 + d_{10})] &= r_4, \\
G[d_9 + d_{10} + \alpha(d_1 + d_7) + \alpha(d_2 + d_8)] &= r_5.
\end{aligned} \tag{13}$$

This refinement differs from the one in the previous Example in that

- a. the five extreme points of \mathcal{C} correspond to five one-dimensional faces of Δ ;
- b. the ten non-trivial extreme points of $\mathcal{P}_{\mathcal{C}}$ correspond to single, non-extreme points of \mathcal{P}_{Δ} ;
- c. the map F is total, it is indeed a parallel projection of Δ onto \mathcal{C} .

This refinement is in fact more similar to that of Example 1, with the difference that the simplex from which the polytope is obtained by projection is not the one of the least possible dimension. \clubsuit

4 Conjectures and questions

I do not know of any study of the general properties of simplicial refinements of a statistical model. Apart from Linusson's theorem, general theorems are lacking.

It seems that if we try to construct a refinement of a statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ with a simplex Δ having fewer extreme points than \mathcal{C} , the construction runs into problems similar to those of Counter-example 2 for the map G . Moreover, even considering higher-dimensional simplices, it seems that if we try to construct $F: \Delta \hookrightarrow \mathcal{C}$ in such a way that only some *non-extreme* points of Δ map onto some extreme points of \mathcal{C} , as in Example 3, then those non-extreme points have to be enough 'far apart' face-wise, otherwise we cannot construct G , again for problems like those in Counter-example 2.

These remarks naturally lead to the following conjectures. Unfortunately I have not been able to prove or disprove them, and wish that convex geometers will take note of them:

Consider a non-simplicial statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$, where \mathcal{C} has m extreme points:

Conjecture 1. All simplicial refinements of $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ have the form $(\Delta, \mathcal{P}_{\Delta}, \bar{F} \circ \pi, \bar{G} \circ \gamma)$, where:

- $(\bar{\Delta}, \mathcal{P}_{\bar{\Delta}}, \pi, \gamma)$ is the refinement of $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ where $\bar{\Delta}$ is the $(m - 1)$ -dimensional simplex from which \mathcal{C} is obtained by parallel projection π , as in Example 1,
- $(\Delta, \mathcal{P}_{\Delta}, \bar{F}, \bar{G})$ is a refinement of $(\Delta, \mathcal{P}_{\Delta})$, where Δ is a simplex of dimension larger than $(m - 1)$ and $\bar{F}: \Delta \hookrightarrow \bar{\Delta}$ is an affine, onto, partial map (a partial projection, an intersection, or a combination of the two).

In other words, the refinement obtained by projection of an $(m - 1)$ -dimensional simplex is the lowest-dimensional one. See fig. 9.

Conjecture 2. No simplicial refinement $(\Delta, \mathcal{P}_{\Delta}, F, G)$ exists with Δ having fewer extreme points than \mathcal{C} . This is also a corollary of the previous conjecture.

Conjecture 3. Let $(\Delta, \mathcal{P}_{\Delta}, F, G)$ be any refinement of $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$, and a, b be any two extreme points of \mathcal{C} . Let $F^{-1}(a), F^{-1}(b)$ be their counter-images in Δ , and A, B the minimal faces of Δ containing these counter-images. Then $A \cap B = \emptyset$. In other words, no refinement exists such that two extreme points of \mathcal{C} correspond to points in Δ lying on adjacent faces.

The following are very important questions for the application of the theory of statistical models to quantum theory and general

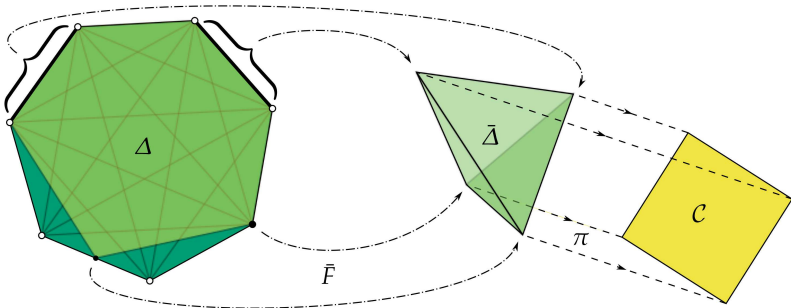


Figure 9: Conjecture 1 says that any refinement $(\Delta, \mathcal{P}_{\Delta}, F, G)$ of $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ has the schema of the above figure, with $F = \bar{F} \circ \pi$ and $G = \bar{G} \circ \gamma$. In other words, the refinement obtained by projection is the one of lowest dimension.

physical statistical theories. Given a non-simplicial statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$:

Question 1. Does a simplicial refinement exist with Δ having fewer extreme points than \mathcal{C} ? (Cf. Conjecture 2.)

Question 2. What is the least dimension of Δ among all simplicial refinements?

Question 3. For each simplicial refinement, can the affine partial map F be extended to a map $\text{aff } \Delta \rightarrow \text{aff } \mathcal{C}$? If so, is the extension unique? What about an analogous extension of G ?

Question 4. How to extend the theory of simplicial refinements to statistical models $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ where \mathcal{C} has a continuum of extreme points or even infinite dimension?

Appendix: Physical motivations

In quantum theory and other physical statistical theories the set of statistical states of a system has the structure of a convex set \mathcal{C} (generally having a continuum of extreme points, and in some cases infinite dimension). By statistical states I mean, e.g., the statistical operators in quantum theory or the Liouville distributions in statistical mechanics. The set of measurement outcomes, for all kinds of measurement that can be made on the system, has the convex structure of the set $\mathcal{P}_{\mathcal{C}}$ (strictly speaking this only holds for *maximal* theories [4], but all known physical theories are maximal). The probability of obtaining the outcome represented by a form $v \in \mathcal{P}_{\mathcal{C}}$ given that the measurement associated to the same form is made on the state represented by a point $s \in \mathcal{C}$ is then given by the action of the form v on the point s , $v \cdot s$. The probabilistic features of a physical system are thus represented by a statistical model $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$. In what follows I assume these notions valid (with topological care) for convex spaces with a continuum of extreme points and possibly infinite dimensions.

The convex structure of the state space \mathcal{C} of quantum and other statistical theories is the origin of many peculiar statistical features, e.g. the fact that some pure states (i.e. states represented by extreme points) cannot be distinguished by one measurement instance, or the fact that no measurement can resolve situations of uncertainties between two or more particular, different sets of pure states. Most clas-

sical systems do not present these peculiarities: because their state spaces have simplicial convex structures Δ .

The connexions between states and measurements of two physical systems are translated into the connexions between the statistical models associated to them. This is a case that has interested physicists for many years: to reproduce the probabilistic features of quantum systems as emergent from those of classical ones; more or less as it happens with thermostatics and statistical mechanics. The quantum system's phenomenology would then be a 'coarser' version of the classical system's one.

This relationship between a 'coarser' physical system represented by $(\mathcal{C}, \mathcal{P}_{\mathcal{C}})$ and a 'finer' classical one represented by $(\Delta, \mathcal{P}_{\Delta})$ implies a correspondence between their states and measurement outcomes with several natural requirements:

- a. all states in the coarser system must have a correspondent in the finer, otherwise the latter system would not be able to describe all the phenomenology of the former;
- b. the finer system can have states with no counterpart in the coarser, because its phenomenology can be richer;
- c. several states in the finer system can correspond to the same state in the coarser: in the coarser system we simply cannot tell them apart;
- d. at most one state in the finer system may correspond to one in the coarser, because the finer system cannot have less 'resolving power' than the coarser;
- e. a situation of uncertainty among states in the finer system must correspond, probability-wise, to the same situation of uncertainty among the corresponding states in the coarser, otherwise the finer system could not reproduce the statistical phenomenology of the coarser;

and analogous requirements hold for the measurement outcomes of the two systems. Moreover, the outcome probabilities for all states and measurement outcomes of the coarser system must obviously be reproduced by the finer one if the statistical phenomenology of the former is to be obtained. We immediately see that these requirements are translated into the characteristics C1–C5 and eq. (5) of § 3 for the statistical models associated to the coarser and finer systems. In other words, we are saying that the statistical model associated to

the finer classical system is a *simplicial refinement* $(\Delta, \mathcal{P}_\Delta, F, G)$ of that associated to the coarser physical system $(\mathcal{C}, \mathcal{P}_\mathcal{C})$.

The best-known example of a classical-like theory reproducing quantum systems described by Schrödinger's equation is Bohmian mechanics [14–16]. Its pure states are of the form (ψ, q) , where the $\{\psi\}$ are in one-one correspondence with quantum pure states (wave functions) and the $\{q\}$ are extra configuration variables. A quantum pure state ψ corresponds in Bohm's theory to the mixed state given by the distribution

$$L(\psi', q') d\psi' dq' := \delta(\psi' - \psi) |\psi(q')|^2 d\psi' dq'; \quad (14)$$

thus the refinement given by Bohm's theory behaves analogously to Example 3 of § 3.

The conjectures of the last section, extended to the case of convex sets with a continuum of extreme points and possibly infinite dimensions, translate into physical requirements for a classical system to be a finer version of generic statistical one, a quantum one in particular:

Physical conjecture 1. Given a generic physical statistical system, e.g. a quantum one, the simplest classical physical system capable of reproducing its statistical phenomenology is one whose set of pure states can be put in one-one correspondence with the set of pure states of the generic one. In the quantum case this says that models like the Beltrametti-Bugajski one [17], which use Holevo's construction [2, § I.5], are the simplest possible.

Physical conjecture 2. If a classical system reproduces the statistical phenomenology of a generic physical statistical, e.g. a quantum one, then its set of pure states cannot have lower cardinality than the generic system's one. In the quantum case, the manifold of pure states of the classical system cannot have less dimensions than that of the quantum one. This means that behind the phenomenology of Schrödinger's equation there must be *physical fields*: particles only are not enough.

This is true of all classical models that reproduce quantum ones. For example, the set of pure states of Bohmian mechanics obviously has a higher cardinality than the quantum one. Also other models, like Nelson's [18; 19], have a greater cardinality than the quantum systems they reproduce, because beside the configuration variables q

they introduce other quantities that can only be interpreted as physical fields from the way they enter into the equations of motion.

Physical conjecture 3. Let ψ_a, ψ_b be two pure states of the generic statistical system (e.g., again, a quantum one), and $F^{-1}(\psi_a), F^{-1}(\psi_b)$ the sets of statistical classical states reproducing those states. Let A, B be the sets of pure classical states into which $F^{-1}(\psi_a), F^{-1}(\psi_b)$ can be purified. Then the sets A, B have neither pure nor mixed states in common.

This conjecture is also satisfied in Nelson’s model or in Bohmian mechanics, owing to the delta function in eq. (14) that makes two distributions corresponding to two different quantum pure states to have disjoint supports. Proving Conjecture 3 would rule out models that Harrigan & Spekkens [20] call ‘ ψ -epistemic’, not only for quantum theory but for *any* non-classical (i.e., non-simplicial) maximal physical statistical theory.

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